Systems of Linear Equations Section 2.1, Goldstein, Schneider and Siegel

(The chapter of the book is available in Sakai under "resources")

The equation of a line is given by:

$$\boxed{\text{ax + by = c}}$$

for numbers a, b and c.

The **Standard form** of the equation of a (non-vertical) line is given by rearranging its equation to the form:

$$\boxed{\text{y = mx + d}}$$

You may be familiar with the fact that the graph of this equation in the xy-plane is a line with slope m and y-intercept d. The graph of the line is the set of all points (x, y) which fit the equation of the line. (The **y-intercept** of a (non-vertical) line is the point where the line crosses the y-axis.)

Example Consider the line

$$2x + 3y = 12$$

Graph this line.

Example Add a graph of the line $3x + 2y = 12$ to your picture above and find the values of x and y at the point where both lines meet.

Solving systems of Linear Equations

An equation of the type $2x + 3y = 12$ is called a linear equation. We see above, that there are infinitely many pairs (x, y) which satisfy this equation.

When we have more than one linear equation, the equations form a system. The equations

$$\begin{cases} 2x + 3y &= 12 \\ 3x + 2y &= 12 \end{cases}$$

form **a system of linear equations.**

A solution to the above system of linear equations is a pair (x_0, y_0) so that (x_0, y_0) fits both equations.

We saw above using a graph that there is exactly one solution to the above system of linear equations, namely the pair $(12/5, 12/5)$ or $x = 12/5, y = 12/5$.

We can also define linear equations and linear systems for more than two variables, however graphing linear systems with 3 variables is difficult and graphing systems with 4 or more variables is impossible. Hence, we must use algebra to solve such systems of equations.

A linear Equation in n variables $\{x_1, x_2, \ldots, x_n\}$ is an equation of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b_1$$

where $a_1, a_2, \ldots, a_n, b_1$ are real numbers.

Example the following is a linear equation in the 3 variables x, y and z

$$2x + 3y + 4z = 9.$$

A **solution to the linear equation**

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b_1$$

is a set of values for $\{x_1, x_2, \ldots, x_n\}$ which fit the equation.

Example The triple $x = 1, y = 1, z = 1$ is a solution to the equation

$$2x + 3y + 4z = 9.$$

The triple $x = 2, y = 1, z = 1/2$ is another solution to the equation.
There are infinitely many solutions to this linear equation. In fact you will find that you can choose two of the three values of x, y and z and the other is determined.
Find another solution to the linear equation

$$2x + 3y + 4z = 9.$$

A **System of m Linear Equations** in n variables $\{x_1, x_2, \ldots, x_n\}$ is a set of linear equations

$$\begin{cases} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n &= b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n &= b_2 \\ \vdots & \vdots \\ a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n &= b_m \end{cases}$$

where $a_{i,j}$, $1 \leq i \leq m$, $1 \leq j \leq n$ and b_i, $1 \leq i \leq n$ are real numbers.

Example A system of three linear equations in the three variables x, y and z is given by

$$\begin{cases} 2x + 3y + 4z &= 9 \\ x + y + z &= 3 \\ x + 5y + 10z &= 16 \end{cases}$$

A **Solution to a system of linear equations** of the type

$$\begin{cases} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n &= b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n &= b_2 \\ \vdots & \vdots \\ a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n &= b_m \end{cases}$$

is a set of values for $\{x_1, x_2, \ldots, x_n\}$ which satisfy all of the equations.

Example Check that $x = 1, y = 1, z = 1$ gives a solution to the set of equations

$$\begin{cases} 2x + 3y + 4z &= 9 \\ x + y + z &= 3 \\ x + 5y + 10z &= 16 \end{cases}$$

A theorem from Linear algebra tells us that there are three possibilities for the set of all solutions to a system of linear equations (as shown above)
1. The solution set may be infinite
2. The solution set may have exactly one elemet (there is exactly one solution)
3. The system of equations has no solutions. (The equations are inconsistent).

In our course, we will consider only situations where the system of equations has exactly one solution. We will solve systems with more than two variables with a computer program called Mathematica.

Example An athlete is planning a diet for a twelve week training plan. For each meal, they have a prescribed balance of carbohydrates, protein and fat. The athlete will use three foods for one particular meal.
One ounce of Food 1 has 30% of the required carbohydrates, 20% of the required protein and .1% of the required fat.
One ounce of Food 2 has 10% of the required carbohydrates, 40% of the required protein and 0% of the required fat.
One ounce of Food 3 has 0% of the required carbohydrates, 0% of the required protein and 50% of the required fat.

Let x, y and z denote the number of ounces of Foods 1,2 and 3 respectively to be included in the meal. What system of linear equations should the athlete solve to find the number of ounces of each food to include in the meal?

Massey method Overview

In a round robin tournament, the point differential is a reasonable way to decide on a winner. Hence given partial results of a round robin tournament, it is reasonable create a ranking for the teams based on the point differential. If we could give each team a rating (upon which we will base our ranking) where the difference between the rating for team i (r_i) and the rating for team j (r_j) is exactly the point differential when team i plays team j, our rating would be ideal.

Example If we have six competitors; as we did in our class tournament, we label the competitors 1 through six. Below we show the results of our class tournament after 2 rounds. We wish to give each competitor a rating. We would like to give competitor i a rating r_i so that for every game that has been already played between competitor i and another competitor; competitor j, we have

$$r_i - r_j = \text{points for competitor i in that game - points for competitor j in that game.}$$

This means that we are looking for a solution to the following system of equations:

Pong Tournament (Round Robin): Results for Rounds 1 and 2

Round 1

Player 1	Emily Aberle	1	vs.	Player 6	Danielle	1
Player 5	Jubril Dawodu	1	vs.	Player 2	Mark Miclean	1
Player 3	Colin Rahill	2	vs.	Player 4	Josh Dunlap	3

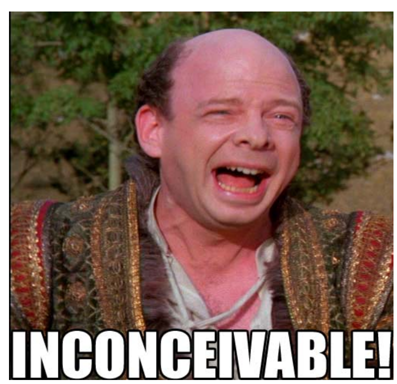
Round 2

Player 5	Jubril Dawodu	1	vs.	Player 3	Colin Rahill	0
Player 2	Mark Miclean	1	vs.	Player 1	Emily Aberle	0
Player 6	Danielle	2	vs.	Player 4	Josh Dunlap	1

We are looking for a set of ratings for the players $\{r_1, r_2, r_3, r_4, r_5, r_6\}$ which are solutions to the system of equations

$$\begin{array}{rcrcr} r_1 & - & r_6 & = & 0 \\ r_5 & - & r_2 & = & 0 \\ r_3 & - & r_4 & = & -1 \\ r_5 & - & r_3 & = & 1 \\ r_2 & - & r_1 & = & 1 \\ r_6 & - & r_4 & = & 1 \end{array}$$

Unfortunately this system of equations does not have a solution, if it did we would have to have : $r_1 = r_6$ and $r_5 = r_2$ from equations 1 and 2 respectively. Equation 3 tells us that $r_4 = r_3 + 1$ and equation 4 tells us that $r_5 = r_3 + 1$, hence we must have $r_4 = r_5 = r_2 = r_3 + 1$. Now the last two equations tell us that $r_2 = r_1 + 1$ and $r_6 = r_4 + 1$. However, $r_2 = r_4$ and $r_1 = r_6$ so we must have $r_4 = r_6 + 1$ and $r_6 = r_4 + 1$. This is impossible!



The Massey rating system, find a rating which is "close" to satisfying the above conditions and which turns out to be a solution to the equations shown below (these equations are sums in pairs of the equations given above):

$$\begin{array}{rcrcrcr} 2r_1 & - & r_2 & - & r_6 & = & -1 \\ 2r_2 & - & r_1 & - & r_5 & = & 1 \\ 2r_3 & - & r_4 & - & r_5 & = & -2 \\ 2r_4 & - & r_3 & - & r_6 & = & 0 \\ 2r_5 & - & r_2 & - & r_3 & = & 1 \\ 2r_6 & - & r_1 & - & r_4 & = & 1 \end{array}$$

There is an equation above for each competitor. For competitor i, the equation is given by :

$$t_i r_i - (\text{the sum of }) n_{ij} r_j = P_i$$

where t_i denotes the number of games player i has played so far, and for $i \neq j$, n_{ij} = the number of times comp. i has played comp. j so far. P_i denotes the point differential (so far) for player i.

This system of equations has infinitely many solutions.

Example Check that both of the following give solutions to the equations:

$$r_1 = -(1/3), r_2 = 1/3, r_3 = -4/3, r_4 = -2/3, r_5 = 0, r_6 = 0.$$

$$r_1 = 2/3, r_2 = 4/3, r_3 = -1/3, r_4 = 1/3, r_5 = 1, r_6 = 1.$$

Note however that both of these ratings give us the same ranking for our competitors.

Emily Aberle 4, Mark Miclean 1, Colin Rahill 6,
Josh Dunlap 5, Jubril Dawodu 2, Danielle 2.

The first five equations contain all of the information about the system above, a sixth equation is superfluous. To find a unique Massey Rating, the last equation is replaced by the condition that all of the ratings sum to 1. That is the Massey ratings for our tournament after round 2 are given by the solution to the following system of equations:

$$\begin{array}{rcrcrcr} 2r_1 & - & r_2 & - & r_6 & = & -1 \\ 2r_2 & - & r_1 & - & r_5 & = & 1 \\ 2r_3 & - & r_4 & - & r_5 & = & -2 \\ 2r_4 & - & r_3 & - & r_6 & = & 0 \\ 2r_5 & - & r_2 & - & r_3 & = & 1 \end{array}$$

$$r_1 + r_2 + r_3 + r_4 + r_5 + r_6 = 1$$

The Massey Rankings are the unique solution to this system

$$r_1 = 1/6, \quad r_2 = 5/6, \quad r_3 = -5/6, \quad r_4 = -1/6, \quad r_5 = 1/2, \quad r_6 = 1/2$$

The Colley Ratings are given by a similar system of equations involving wins minus losses.